Annals of Mathematics, **155** (2002), 915–928

Hodge structures on abelian varieties of type III

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Abstract

We show that the usual Hodge conjecture implies the general Hodge conjecture for certain abelian varieties of type III, and use this to deduce the general Hodge conjecture for all powers of certain 4-dimensional abelian varieties of type III. We also show the existence of a Hodge structure M such that M occurs in the cohomology of an abelian variety, but the Tate twist $M(1)$ does not occur in the cohomology of any abelian variety, even though it is effective.

1. Introduction

Consider a Hodge structure $V_{\mathbf{C}} = \oplus_{p+q=n} V^{p,q}$ of weight n. The Hodge structure is said to be *effective* if $V^{p,q} = 0$ unless $p, q \geq 0$. It is said to be *geometric* if it is isomorphic to a Hodge substructure of the cohomology of a smooth projective algebraic variety over $\mathbf{C}$. For an integer m, the *Tate twist* $V(m)$ is the Hodge structure of weight $n-2m$ defined by $V(m)^{p,q} := V^{p+m,q+m}$.

The general Hodge conjecture as formulated by Grothendieck [9] implies that any Tate twist of a geometric Hodge structure is again geometric, provided that it is effective. We say that a smooth projective variety X is *dominated* by a class of varieties $\mathcal{Y}$ if, for every irreducible Hodge structure M occurring in the cohomology of X, the Tate twist $M(h)$ occurs in the cohomology of some member of $\mathcal{Y}$, where

$$h := \min\{p \mid M^{p,n-p} \neq 0\}$$

is the *height* of the Hodge structure. Grothendieck observed that the general Hodge conjecture for X is implied by the existence of a class of varieties $\mathcal{Y}$ which dominates X, together with the usual Hodge conjecture for $Y \times X$ for each $Y \in \mathcal{Y}$ (cf. [2, Proposition 2.1, p. 343]).

In a series of papers ([2], [4], [5]) we showed that certain abelian varieties are dominated by subclasses of the class of all abelian varieties, and used this to deduce the general Hodge conjecture in some cases. This raised the question of

*2000 *Mathematics Subject Classification.* Primary 14C30, 14K20; Secondary 20G30.

whether every abelian variety is dominated by the class of all abelian varieties. We answer this question in the negative in this paper (Theorem 5.5).

Let A be a simple abelian variety of type III. Recall that A is said to be of type III if its endomorphism algebra $D := \mathrm{End}(A) \otimes \mathbf{Q}$ is a totally definite quaternion algebra over a totally real number field. Assume that A is a general member of a PEL-family. In [2, Theorem 5.1, p. 348] we showed that A is dominated by the set of powers of itself, provided that $m := \dim_D H_1(A, \mathbf{Q})$ is odd. Using this we were able to prove the general Hodge conjecture for all powers of certain 6-dimensional abelian varieties of type III in [3, Example 5.1, p. 674]. In this paper we show (Theorem 4.1 and Proposition 5.1) that A is dominated by the powers of itself if and only if $\operatorname{disc} T$, the discriminant of the D-valued skewhermitian form determined by a polarization of A, is not a square. As a consequence we prove the general Hodge conjecture for all powers of certain 4-dimensional abelian varieties of type III (Corollary 4.3).

When $\operatorname{disc} T$ is a square and $m = 4$, we show that there exists an abelian variety B such that A is dominated by abelian varieties of the form $A^i \times B^j$ (Theorem 5.3). However, when $\operatorname{disc} T$ is a square and $m > 4$, we show that there exists a Hodge structure $M \subset H^m(A, \mathbf{Q})$ such that $M(1)$ is effective but does not occur in the cohomology of any abelian variety (Theorem 5.5). The least dimension of the abelian varieties in which these extraordinary Hodge structures occur is 12, in which case $M \subset H^6(A, \mathbf{Q})$. Weil [24] proposed that certain Hodge cycles be considered as possible counterexamples to the usual Hodge conjecture. In the same spirit, we suggest that these extraordinary Hodge structures be used as a testing ground for the general Hodge conjecture.

Acknowledgements. Lemma 2.1.1 was formulated during a discussion with Chad Schoen at Banff in June 1998. I thank him for this and other suggestions. I thank the referees for their careful reading of the manuscript; their comments have helped improve the paper.

Notation and conventions. All abelian varieties are over $\mathbf{C}$. For an abelian variety A, we let $D(A) := \mathrm{End}(A) \otimes \mathbf{Q}$ be the endomorphism algebra of A, we denote by $G(A)$ the Hodge group of A (§2.1), and we write $\mathfrak{g}(A)$ for the Lie algebra of $G(A)$. The semisimple part (derived group) of $G(A)$ is denoted by $G^{\mathrm{ss}}(A)$, and $\mathfrak{g}^{\mathrm{ss}}(A) := \operatorname{Lie} G^{\mathrm{ss}}(A)$ is its Lie algebra.

The definition of *dominated* given in this paper is slightly different from that given in [2, p. 344]; the two definitions are equivalent if the general Hodge conjecture is true.

2. Abelian varieties and Hodge groups

2.1. *Hodge groups.* By a Hodge structure of weight n, we mean a finite-dimensional vector space V over $\mathbf{Q}$, and a direct sum decomposition $V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$. Let $\mathbf{S}$ be the real algebraic group such

that $\mathbf{S}(\mathbf{R})$ is the unit circle in the complex plane. Then a Hodge structure V determines a morphism

$$\tau_V : \mathbf{S} \to \mathrm{GL}(V_{\mathbf{R}})$$

of real algebraic groups such that $\tau_V(e^{i\theta})$ acts as multiplication by $e^{(p-q)i\theta}$ on $V^{p,q}$. The *Hodge group* of V is defined to be the smallest $\mathbf{Q}$-subgroup, G, of $\mathrm{GL}(V)$ such that $G_{\mathbf{R}}$ contains the image of τ_V.

The *Hodge group*, $G(A)$, of an abelian variety A is defined to be the Hodge group of $V := H^1(A, \mathbf{Q})$. A Riemann form for A gives an isomorphism of V with $V^\star$ which is a morphism of Hodge structures, so the Hodge group of A may also be defined as the Hodge group of $H_1(A, \mathbf{Q}) \cong H^1(A, \mathbf{Q})^\star$. The Hodge group of an abelian variety is a connected, reductive group of hermitian type which has no nontrivial, connected, normal $\mathbf{Q}$-subgroup H such that $H(\mathbf{R})$ is compact ([12], [6], [8]).

LEMMA 2.1.1. *Let M be a Hodge structure contained in the cohomology of an abelian variety A. Then the Hodge group of M is a quotient of the Hodge group of A.*

Proof. Let G be the Hodge group of A, H the Hodge group of M, and, $V := H^1(A, \mathbf{Q})$. Since M is a G-submodule of the exterior algebra of V, we have a morphism $\varphi : G \to \mathrm{GL}(M)$. This is compatible with the Hodge structures in the sense that $\varphi(\tau_V(g)) = \tau_M(g)$ for all $g \in \mathbf{S}(\mathbf{R})$. Therefore $\varphi(G)$ contains the image of τ_M, and hence contains H. Since $\varphi^{-1}(H)$ is a $\mathbf{Q}$-subgroup of G containing the image of τ_V, we see that $\varphi^{-1}(H) = G$. Thus $H = \varphi(G)$. $\square$

2.2. *Kuga fiber varieties.* A general reference for this subsection is Satake's book [17].

Let G be a connected, semisimple, linear algebraic group over $\mathbf{Q}$. Assume that G is of hermitian type, and has no nontrivial, connected, normal $\mathbf{Q}$-subgroup H such that $H(\mathbf{R})$ is compact. Then $X := G(\mathbf{R})^0/K$ is a bounded symmetric domain, where K is a maximal compact subgroup of $G(\mathbf{R})^0$. Let $\mathfrak{g} := \mathrm{Lie}\, G$ be the Lie algebra of G, $\mathfrak{k} := \mathrm{Lie}\, K$, and let $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let o be the unique fixed point of K in X. Differentiating the natural map $G(\mathbf{R})^0 \to X$ gives an isomorphism of $\mathfrak{p}$ with $T_o(X)$, the tangent space of X at o, and there exists a unique $H_0 \in Z(\mathfrak{k})$, called the *H-element* at o, such that $\mathrm{ad}\, H_0|\mathfrak{p}$ is the complex structure on $T_o(X)$.

Let β be a nondegenerate alternating form on a finite-dimensional rational vector space V. The symplectic group $\mathrm{Sp}(V, \beta)$ is a $\mathbf{Q}$-algebraic group of hermitian type; the associated symmetric domain is the *Siegel space*

$$\begin{aligned} \mathfrak{S}(V,\beta) \quad := \quad \{J \in \mathrm{GL}(V_{\mathbf{R}}) \mid J^2 = -I \text{ and} \\ \beta(x, Jy) \text{ is symmetric, positive definite}\}. \end{aligned}$$

$\mathrm{Sp}(V,\beta)$ acts on $\mathfrak{S}(V,\beta)$ by conjugation. The H-element at a point $J \in \mathfrak{S}(V,\beta)$ is $J/2$.

Let $\rho : G \to \mathrm{Sp}(V,\beta)$ be a representation defined over $\mathbf{Q}$. We say that ρ satisfies the H_1-*condition* relative to the H-elements H_0 and $H_0' = J/2$ if

$$[d\rho(H_0) - H_0', d\rho(g)] = 0 \qquad \text{for all } g \in \mathfrak{g}_{\mathbf{R}},$$

and we say that ρ satisfies the H_2-*condition* if

$$d\rho(H_0) = H_0'.$$

Assume that the H_1-condition is satisfied relative to H_0 and H_0'. Then there exists a unique holomorphic map $\tau : X \to \mathfrak{S}(V,\beta)$ such that $\tau(o) = J$, and the pair (ρ,τ) is equivariant in the sense that

$$\tau(g\cdot x) = \rho(g)\cdot\tau(x) \qquad \text{for all } g \in G(\mathbf{R})^0, x \in X.$$

Let Γ be a torsion-free arithmetic subgroup of $G(\mathbf{Q})$. Let L be a lattice in V such that $\rho(\Gamma)L = L$. The quotient

$$\mathcal{A} := (\Gamma \ltimes_\rho L)\backslash(X \times V_{\mathbf{R}})$$

is a smooth quasiprojective algebraic variety (see [6, p. 74]); the natural projection $\mathcal{A} \to \mathcal{V} := \Gamma\backslash X$ makes $\mathcal{A}$ a fiber variety over $\mathcal{V}$ called a *Kuga fiber variety*. The fiber $\mathcal{A}_P$ over any point $P \in \mathcal{V}$ is an abelian variety isomorphic to the torus $V_{\mathbf{R}}/L$ with the complex structure $\tau(x)$, where x is a point in X lying over P.

LEMMA 2.2.1. *Let $\rho_j : G \to \mathrm{Sp}(V_j,\beta_j)$, $j = 1,2$, be H_2-representations defining Kuga fiber varieties $\mathcal{A}_j \to \mathcal{V}$. Let $P \in \mathcal{V}$. If ρ_1 and ρ_2 are equivalent as representations of G, then, the abelian varieties $\mathcal{A}_{1,P}$ and $\mathcal{A}_{2,P}$ are isogenous.*

Proof. Let $\psi : V_1 \to V_2$ be an equivalence, and $\psi_\star$ the induced isomorphism from $\mathrm{GL}(V_1)$ to $\mathrm{GL}(V_2)$. The abelian variety $\mathcal{A}_{j,P}$ is the torus $V_{j,\mathbf{R}}/L_j$ with the complex structure J_j, where L_j is a lattice in V_j, and $J_j = 2d\rho_j(H_0)$. Since $\psi_\star(J_1) = J_2$, ψ induces an isogeny from $\mathcal{A}_{1,P}$ to $\mathcal{A}_{2,P}$. ☐

2.3. *Hodge families.* Given an abelian variety A, Mumford [12] constructed a Kuga fiber variety having A as a general fiber. These Kuga fiber varieties are called *Hodge families*, and have interpretations as solutions to fine moduli problems. We briefly review Mumford's construction here.

Let A be an abelian variety, and $V := H_1(A,\mathbf{Q})$. As a complex torus, A is isomorphic to $V_{\mathbf{R}}/V_{\mathbf{Z}}$ with a complex structure J, where $V_{\mathbf{Z}}$ is the lattice $H_1(A,\mathbf{Z})$. A polarization of A determines a Riemann form, i.e., an alternating form, β, on V such that $\beta(V_{\mathbf{Z}},V_{\mathbf{Z}}) \subset \mathbf{Z}$, and $\beta(x,Jy)$ is symmetric and positive definite. Let X be the symmetric domain belonging to $G^{\mathrm{ss}}(A)$, and choose a base point $o \in X$. The inclusion

$$\rho : G^{\mathrm{ss}}(A) \to \mathrm{Sp}(V,\beta)$$

satisfies the H_1-condition with respect to the H-elements at o and J. Let Γ be a torsion-free arithmetic subgroup of $G^{\mathrm{ss}}(A)$ such that $\rho(\Gamma)(V_{\mathbf{Z}}) \subset V_{\mathbf{Z}}$. The Kuga fiber variety $\mathcal{A} \to \mathcal{V}$ obtained from this data is called the *Hodge family determined by* A. If P is the image of o in $\mathcal{V}$, then the fiber $\mathcal{A}_P$ is isomorphic to A, and the Hodge group of every member of the family is contained in $G(A)$.

Proposition 2.3.1 ([1, Prop. 2.2, p. 1124]). *Let G be a connected, semisimple, linear algebraic group over $\mathbf{Q}$ of hermitian type. Assume that G has no nontrivial, connected, normal $\mathbf{Q}$-subgroup H such that $H(\mathbf{R})$ is compact. Any Kuga fiber variety $\mathcal{A} \to \mathcal{V}$ constructed from a symplectic representation $\rho : G \to \mathrm{Sp}(V, \beta)$ satisfying the H_2-condition, is a Hodge family, and the Hodge group of a general fiber equals $\rho(G)$.*

2.4. *Primary representations.* Recall that a fully reducible representation is called *primary* if all irreducible representations contained in it are equivalent. Let $\rho : G \to \mathrm{Sp}(V, \beta)$ be an H_1-representation of a semisimple hermitian group G over $\mathbf{Q}$. We have the primary decomposition $V = \oplus_\alpha V_\alpha$, where the V_α are the maximal primary G-submodules of V (cf. Satake [17, p. 167]). Each representation

$$G \to \mathrm{Sp}(V_\alpha, \beta | V_\alpha \times V_\alpha)$$

satisfies the H_1-condition (see Satake [17, Lemma IV.4.2, p. 181] and the remarks following it).

Lemma 2.4.1. *Let $\rho : G \to \mathrm{Sp}(V, \beta)$ be a primary representation satisfying the H_1-condition, and defined over $\mathbf{Q}$. Suppose $G = G_1 \times G_2$, and $G_1(\mathbf{R})$ has no compact factors. Then either G_1 or G_2 acts trivially on V.*

Proof. Suppose both G_1 and G_2 act nontrivially on V. Let U be an irreducible submodule of $V_{\overline{\mathbf{Q}}}$. Then U^τ is also an irreducible submodule of $V_{\overline{\mathbf{Q}}}$ for any automorphism τ of $\overline{\mathbf{Q}}$, and we have $V_{\overline{\mathbf{Q}}}$ equal, up to multiplicity, to the sum of the distinct $\mathrm{Aut}(\overline{\mathbf{Q}})$-conjugates of U (see Satake [16, §1.1, pp. 218–221]). If G_1 (resp. G_2) acts trivially on U, then G_1 (resp. G_2) acts trivially on every conjugate of U, so we infer that both G_1 and G_2 act nontrivially on U.

Some simple factor H of G_2 acts nontrivially on U. We have $H = R_{K/\mathbf{Q}}\widehat{H}$, where K is a totally real number field, $\widehat{H}$ is an absolutely simple group over K, and, $R_{K/\mathbf{Q}}$ means restriction of scalars from K to $\mathbf{Q}$ (see Satake [16, p. 259]). Let S be the set of embeddings of K into $\mathbf{R}$, let L be the Galois closure of K, and let $\mathcal{G} := \mathrm{Gal}(L/\mathbf{Q})$. Then $\mathcal{G}$ acts transitively on S. We have $H_L = \prod_{a \in S} H_a$, where $H_a := \widehat{H} \otimes_{K,a} L$. Let

$$S_0 := \{a \in S \mid H_a(\mathbf{R}) \text{ is not compact}\}.$$

One of the factors, say H_a, acts nontrivially on U. Since $\mathcal{G}$ acts transitively on S, there exists $\sigma \in \mathcal{G}$ such that $a^\sigma = b$, and, $b \in S_0$. Extend σ to an

automorphism of $\overline{\mathbf{Q}}$, and denote the extension again by σ. Then U^σ is an irreducible subrepresentation of $V_{\overline{\mathbf{Q}}}$ such that both G_1 and H_b act nontrivially on it.

Let $V_{\mathbf{R}} = \oplus_\alpha V_\alpha$ be the primary decomposition of $V_{\mathbf{R}}$ as a $G_{\mathbf{R}}$-module. Then we have $V_{\mathbf{C}} = \oplus_\alpha V_{\alpha,\mathbf{C}}$. Since $U^\sigma_{\mathbf{C}}$ is an irreducible submodule of $V_{\mathbf{C}}$, there exists α such that $V_{\alpha,\mathbf{C}}$ contains a submodule equivalent to $U^\sigma_{\mathbf{C}}$. Then both G_1 and H_b act nontrivially on V_α. Since $G_1(\mathbf{R})$ has no compact factors, and $H_b(\mathbf{R})$ is not compact, this contradicts Satake's result that each $\mathbf{R}$-primary subrepresentation of $\rho_{\mathbf{R}}$ can be nontrivial on only one noncompact factor of $G(\mathbf{R})$ (Satake [17, pp. 185–186]). □

3. An algebraic group and its Lie algebra

3.1. *Quaternionic skewhermitian forms.* Let k be a totally real number field, D a totally definite quaternion algebra over k, and $x \mapsto \bar{x}$ the canonical involution of D. Let V be a left vector space of dimension m over D. Let T be a D-valued skewhermitian form on V, i.e., a k-bilinear form $T : V \times V \to D$ such that $T(ax, by) = aT(x,y)\bar{b}$ and $T(y,x) = -\overline{T(x,y)}$ for all $a, b \in D$ and all $x, y \in V$. We assume that T is nondegenerate. Recall that the *determinant* $\det T$ is the reduced norm from $M_n(D)$ to k of a matrix representation of T, and the *discriminant* of T is $\operatorname{disc} T = (-1)^m \det T$. The determinant and discriminant are only well-defined modulo squares; i.e., they are elements of $k^\times/k^{\times 2}$.

Example 3.1.1. Recall that an element $\lambda \in D$ is called a *pure* quaternion if $\bar{\lambda} = -\lambda$. The reduced norm of such an element is given by $N(\lambda) = -\lambda^2$. Let $\lambda_1, \dots, \lambda_m$ be nonzero pure quaternions in D. Then $\operatorname{diag}(\lambda_1, \dots, \lambda_m)$ is the matrix of a skewhermitian form T on $V = D^m$. We have $\operatorname{disc} T = \prod_i \lambda_i^2$.

Example 3.1.2. As a special case of the previous example, suppose $\lambda = \lambda_1 = \cdots = \lambda_m$. Then $\operatorname{disc} T = \lambda^{2m}$ is a square in k if and only if m is even.

Example 3.1.3. As another special case of Example 3.1.1, let D be the quaternion algebra over $\mathbf{Q}$ with basis $\{1, i, j, k\}$, where $i^2 = -1$, $j^2 = -3$, and, $ij = k$. Take $m \geq 2$, let $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = i$, and let $\lambda_m = j$. Then $\operatorname{disc} T = \pm 3$ is not a square.

3.2. *An algebraic group.* Let $\widehat{G}$ be the connected component of the identity in the group $\operatorname{Aut}_D(V,T)$. Then $\widehat{G}$ is an algebraic group over k of type D_m. It is semisimple for $m \geq 2$, and absolutely simple for $m > 2$.

Let G be the group $R_{k/\mathbf{Q}}\widehat{G}$ obtained by restricting scalars from k to $\mathbf{Q}$. It is a semisimple algebraic group over $\mathbf{Q}$ for $m \geq 2$, and is $\mathbf{Q}$-simple for $m > 2$.

Let S be the set of embeddings of k into $\mathbf{R}$. Then the real Lie group $G(\mathbf{R})$ is given by

$$G(\mathbf{R}) \cong \prod_{\alpha \in S} G_\alpha \tag{3.2.1}$$

where G_α is the group of real points of $\widehat{G} \otimes_{k,\alpha} \mathbf{R}$. Each G_α is isomorphic to the group of type D_m called $\mathrm{SO}^\star(2m)$ by Helgason [10, p. 445] and $\mathrm{SU}^-(m, \mathbf{H})$ by Satake (see [17, Exercise 2, p. 278]). The real rank of G_α is $[m/2]$. It has Tits index ${}^2D^{(2)}_{m,(m-1)/2}$ if m is odd, and Tits index ${}^1D^{(2)}_{m,m/2}$ if m is even [22, pp. 56–57].

3.3. *The Lie algebra.* Let α be an embedding of k into $\mathbf{R}$. We now recall the description of the Lie algebra $\mathfrak{g}_\alpha$ of G_α, and its representations, following Satake [15, §3.3, pp. 449–451] and Varadarajan [23, p. 329]. We have

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} X_1 & X_{12} \\ -\overline{X}_{12} & \overline{X}_1 \end{pmatrix} \in \mathfrak{sl}_{2m}(\mathbf{C}) \;\middle|\; \begin{array}{c} X_1, X_{12} \in M_m(\mathbf{C}) \\ {}^t\overline{X}_1 = -X_1, {}^tX_{12} = -X_{12} \end{array} \right\},$$

a Cartan subalgebra is given by

$$\mathfrak{h} = \{\mathrm{diag}(a_1, \dots, a_m, -a_1, \dots, -a_m) \in \mathfrak{sl}_{2m}(\mathbf{C}) \mid a_j \in i\mathbf{R}\},$$

and an H-element is given by

$$H_0 = \begin{pmatrix} \frac{i}{2} I_m & 0 \\ 0 & \frac{-i}{2} I_m \end{pmatrix}.$$

Define $\lambda_j : \mathfrak{h}_{\mathbf{C}} \to \mathbf{C}$ by $\lambda_j(\mathrm{diag}(a_1, \dots, a_{2m})) = a_j$. A simple system of roots of $\mathfrak{g}_{\alpha,\mathbf{C}}$ is given by $\Delta = \{\alpha_1, \dots, \alpha_m\}$, where $\alpha_j = \lambda_j - \lambda_{j+1}$ for $1 \le j \le m-1$, and, $\alpha_m = \lambda_{m-1} + \lambda_m$.

The fundamental weights of $\mathfrak{g}_{\alpha,\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$ are $\mu_1, \dots, \mu_m$, where

$$\begin{aligned} \mu_j &= \lambda_1 + \cdots + \lambda_j \qquad \text{for } 1 \le j \le m-2, \\ \mu_{m-1} &= (\lambda_1 + \cdots + \lambda_{m-1} - \lambda_m)/2, \\ \mu_m &= (\lambda_1 + \cdots + \lambda_{m-1} + \lambda_m)/2. \end{aligned}$$

Let W be the standard representation of $\mathfrak{g}_{\alpha,\mathbf{C}}$. Then $\bigwedge^j W$ is an irreducible $\mathfrak{g}_{\alpha,\mathbf{C}}$-module with highest weight μ_j for $1 \le j \le m-2$, $\bigwedge^{m-1} W$ is an irreducible $\mathfrak{g}_{\alpha,\mathbf{C}}$-module with highest weight $\mu_{m-1} + \mu_m$, while $\bigwedge^m W$ splits as the direct sum of two irreducible $\mathfrak{g}_{\alpha,\mathbf{C}}$-submodules, U_1 and U_2, having highest weights $2\mu_{m-1}$ and $2\mu_m$, respectively [23, Chap. 4, Exercise 23, p. 394].

LEMMA 3.3.1. *Let* $\mathrm{SO}_{2m}(\mathbf{C}) \to \mathrm{GL}(W)$ *be the standard representation of* $\mathrm{SO}_{2m}(\mathbf{C})$, *where* $m \ge 2$. *Then, the irreducible representations with highest weights* $2\mu_{m-1}$ *and* $2\mu_m$ *do not occur in* $\bigwedge^{m-2}(W^n)$, *for any positive integer* n.

Proof. An irreducible representation which occurs in $\bigwedge^{m-2}(W^n)$ must occur in $W^{\otimes m-2}$. The set of weights of the standard representation of $\mathfrak{so}_{2m}(\mathbf{C})$ on W is

$$\Lambda = \{\lambda_1, \dots, \lambda_m, -\lambda_1, \dots, -\lambda_m\}.$$

The weights of the representation $W^{\otimes m-2}$ are of the form $a_1 + \cdots + a_{m-2}$, where $a_j \in \Lambda$. Therefore

$$2\mu_{m-1} = \lambda_1 + \cdots + \lambda_{m-1} - \lambda_m$$

and

$$2\mu_m = \lambda_1 + \cdots + \lambda_{m-1} + \lambda_m$$

are not weights of $W^{\otimes m-2}$. □

4. Abelian varieties of type III

Let A be a simple abelian variety, and $D := D(A) := \mathrm{End}(A) \otimes \mathbf{Q}$ its endomorphism algebra. We assume in this section that A is of type III; i.e., D is a totally definite quaternion algebra over a totally real field k. Let $V := H_1(A, \mathbf{Q})$. Then V is a left vector space over D; let m be its dimension. By Shimura [19, Proposition 15, p. 177] we have $m \geq 2$.

Let $\beta : V \times V \to \mathbf{Q}$ be a Riemann form for A. There exists a unique D-valued skewhermitian form T on V such that $\beta(x, y) = \mathrm{Tr}_{D/\mathbf{Q}} T(x, y)$ for all $x, y \in V$, where $\mathrm{Tr}_{D/\mathbf{Q}}$ is the reduced trace from D to $\mathbf{Q}$ (Shimura [20, Lemma 1.2, p. 162]). The group $\widehat{G} := \mathrm{Aut}_D(V, T)^0$ is then of the type discussed in §3.2, and we shall henceforth use the notations introduced in §3. The group G is a semisimple group of hermitian type, and its inclusion in $\mathrm{Sp}(V, \beta)$ satisfies the H_2-condition. The resulting Kuga fiber variety is a PEL-family in the sense of Shimura [21]. For proofs of these facts, and more details, we refer the reader to Satake [17, Example IV.7.2, p. 200], and to [3].

THEOREM 4.1. *Let A be a simple abelian variety of type* III. *Assume that A is a general member of a* PEL-*family. Let T be the skewhermitian form determined by a polarization of A. If the discriminant of T is not a square, then any power of A is dominated by the set of powers of A, and the usual Hodge conjecture for A implies the general Hodge conjecture for all powers of A.*

Proof. Proposition 2.3.1 implies that the Hodge group of A is G. We shall now explicitly describe the Hodge structure on the cohomology of A, in terms of the action of G on V (see also the proof of [2, Theorem 5.1, pp. 348–353]). Corresponding to the decomposition (3.2.1) of $G(\mathbf{R})$, we have $V_{\mathbf{R}} \cong \oplus_{\alpha \in S} V_\alpha$,

where each V_α is a real Hodge substructure of $V_{\mathbf{R}}$ such that G_β acts trivially on V_α unless $\alpha = \beta$. The action of $\mathfrak{g}_\alpha$ on V_α is given by

$$\begin{pmatrix} X_1 & X_{12} \\ -\overline{X}_{12} & \overline{X}_1 \end{pmatrix} \mapsto \begin{pmatrix} X_1 & 0 & 0 & -X_{12} \\ 0 & X_1 & X_{12} & 0 \\ 0 & -\overline{X}_{12} & \overline{X}_1 & 0 \\ \overline{X}_{12} & 0 & 0 & \overline{X}_1 \end{pmatrix}$$

(Satake [15, (17), p. 433 and §3.3, pp. 449–451]) with respect to a suitable basis of $V_{\alpha,\mathbf{C}}$, which we call

$$\mathcal{B} = \{v_{m+1}, \ldots, v_{2m}, u_1, \ldots, u_{2m}, v_1, \ldots, v_m\}.$$

This notation is somewhat perverse, but has been chosen to simplify some of the expressions later in the proof. We observe that

$$V_{\alpha,\mathbf{C}} = W_\alpha \oplus \overline{W}_\alpha,$$

where W_α has basis $\{u_1, \ldots, u_{2m}\}$, and its complex conjugate, $\overline{W}_\alpha$, has basis $\{v_{m+1}, \ldots, v_{2m}, v_1, \ldots, v_m\}$. The action of $\mathfrak{g}_{\alpha,\mathbf{C}} \cong \mathfrak{so}_{2m}(\mathbf{C})$ on both W_α and $\overline{W}_\alpha$ is equivalent to the standard representation.

With respect to the basis $\mathcal{B}$, the complex structure on V_α is given by

$$J = 2H_0 = \mathrm{diag}(iI_{2m}, -iI_{2m})$$

(Satake [15, p. 431]). From this we see that the Hodge structure of V_α is

$$V_{\alpha,\mathbf{C}} = V^{1,0} \oplus V^{0,1},$$

where $V^{1,0}$ has the basis $\{v_{m+1}, \ldots, v_{2m}, u_1, \ldots, u_m\}$, and, $V^{0,1}$ has the basis $\{u_{m+1}, \ldots, u_{2m}, v_1, \ldots, v_m\}$. It follows that

$$W_\alpha = W_\alpha^{1,0} \oplus W_\alpha^{0,1},$$

where $W_\alpha^{1,0}$ has basis $\{u_1, \ldots, u_m\}$ and consists of $(1,0)$-forms, while $W_\alpha^{0,1}$ has basis $\{u_{m+1}, \ldots, u_{2m}\}$ and consists of $(0,1)$-forms.

For $1 \leq j \leq m-2$, $\bigwedge^j W_\alpha$ is an irreducible $G_\alpha(\mathbf{C})$-module with highest weight μ_j. It contains the $(j,0)$-form $u_1 \wedge \cdots \wedge u_j$. The irreducible $G_\alpha(\mathbf{C})$-module $\bigwedge^{m-1} W_\alpha$ has highest weight $\mu_{m-1}+\mu_m$. It contains the $(m-1,0)$-form $u_1 \wedge \cdots \wedge u_{m-1}$.

$\bigwedge^m W_\alpha$ splits as the direct sum of two irreducible $\mathfrak{g}_{\alpha,\mathbf{C}}$-submodules, $U_{1,\alpha}$ and $U_{2,\alpha}$, having highest weights $2\mu_{m-1}$ and $2\mu_m$, respectively. The space of $(m,0)$-forms in $\bigwedge^m W_\alpha$ is the 1-dimensional vector space $\bigwedge^m W_\alpha^{1,0}$ spanned by

$$u := u_1 \wedge \cdots \wedge u_m.$$

Since u belongs to $U_{2,\alpha}$ (it is a vector of highest weight), we conclude that $U_{2,\alpha}$ contains an $(m,0)$-form, while $U_{1,\alpha}$ contains no $(m,0)$-forms.

Since $\operatorname{disc} T$ is not a square, the Tits index of $\widehat{G}$ is ${}^2D^{(2)}_{m,r}$, where r is the k-rank of $\widehat{G}$ [22, p. 57]. In this case, $\mathrm{Gal}(\bar{k}/k)$ transposes the roots α_{m-1} and α_m, while leaving $\alpha_1, \ldots, \alpha_{m-2}$ invariant. This implies that the weights $2\mu_{m-1}$ and $2\mu_m$ belong to the same orbit of $\mathrm{Gal}(\bar{k}/k)$. Therefore, if U is a G-module such that $U_{\mathbf{C}}$ contains $U_{1,\alpha}$, then $U_{\mathbf{C}}$ must contain a $G_\alpha(\mathbf{C})$-module equivalent to $U_{2,\alpha}$. It follows that if U is the smallest Hodge structure containing $U_{1,\alpha}$, then $U_{\mathbf{C}}$ contains $(m, 0)$-forms. We may then proceed as in the proof of Case 3 of [2, Theorem 5.1, pp. 352–353] to show that any power of A is dominated by the set of all powers of A, and hence the usual Hodge conjecture for all powers of A implies the general Hodge conjecture for all powers of A. To complete the proof we note that by [3, Theorem 4.1, p. 673] the usual Hodge conjecture for A implies the usual Hodge conjecture for all powers of A. □

Remark 4.2. Murty [14, Theorem 3.2, p. 203] has shown that the Hodge ring of an abelian variety of type III is never generated by divisors. In [3, Theorem 4.1, p. 673] we showed that if A is a sufficiently general abelian variety of type III, then the Hodge ring of any power of A is generated by divisors and Weil cycles. Using Schoen's results [18] on the algebraicity of certain Weil cycles, it is then possible to prove the usual Hodge conjecture for all powers of certain type III abelian varieties of dimensions 4 [3, Example 5.2, p. 674], 6 [3, Example 5.1, p. 674], and, 8 (van Geemen and Verra [7]). In the 6-dimensional case we were able to deduce the general Hodge conjecture. In the next corollary we do the same for the 4-dimensional case. The general Hodge conjecture in the 8-dimensional case remains open.

COROLLARY 4.3. *If A is a simple 4-dimensional abelian variety of type* III *whose endomorphism algebra contains a square root of* -3, *then, the general Hodge conjecture is true for all powers of A.*

Proof. By Moonen and Zarhin [11, 6.1, p. 568], A is a general member of a PEL-family. By Shimura [19, Proposition 17, p. 180], the discriminant of the skewhermitian form determined by a polarization of A is not a square. As in [3, Example 5.2, p. 674], the usual Hodge conjecture for A follows from [3, Theorem 4.1, p. 673], together with Schoen's proof of the algebraicity of the required Weil cycles [18]. Theorem 4.1 now implies the general Hodge conjecture for all powers of A. □

5. An extraordinary Hodge structure

Throughout this section we retain the notations introduced in previous sections, including the proof of Theorem 4.1. First, we prove a converse to Theorem 4.1.

Proposition 5.1. *Let A be a simple abelian variety of type* III. *Assume that A is a general member of a* PEL-*family. Let T be the skewhermitian form determined by a polarization of A. If the discriminant of T is a square, then A is not dominated by the set of powers of itself.*

Proof. The Tits index of $\widehat{G}$ is ${}^1D^{(2)}_{m,r}$ [22, pp. 56–57], where r is the k-rank of $\widehat{G}$. In this case the $\star$-action (or Δ-action) of $\mathrm{Gal}(\overline{k}/k)$ on the simple roots of $\widehat{G}$ is trivial. By the arguments of Satake [16, §1.1, pp. 218–220] this implies that given any irreducible representation ρ of $\widehat{G}_{\overline{k}}$, some multiple $n\rho$ of ρ is defined over k. Therefore there exists an irreducible representation $\widehat{G} \to \mathrm{GL}(\widehat{U}_1)$ equivalent to a multiple of the representation with highest weight $2\mu_{m-1}$. Restricting scalars to $\mathbf{Q}$, we obtain an irreducible representation χ of G which is equivalent over $\mathbf{C}$ to a multiple of $\oplus_{\alpha\in S} U_{1,\alpha}$.

Choose $\alpha \in S$, and let M be the smallest Hodge substructure of $\bigwedge^m V = H^m(A, \mathbf{Q})$ such that $M_{\mathbf{C}}$ contains $U_{1,\alpha}$. Then the representation of G on M is equivalent to χ. We have seen in the proof of Theorem 4.1 that $U_{1,\alpha}$ contains no $(m, 0)$-forms, for any $\alpha \in S$. Therefore $M_{\mathbf{C}}$ contains no $(m, 0)$-forms, and $M(1)$ is an effective Hodge structure. By Lemma 3.3.1, the representation of $G_\alpha(\mathbf{C})$ with highest weight $2\mu_{m-1}$ does not occur in $\bigwedge^{m-2}(W^n_\alpha)$ for any positive integer n, so $M(1)$ does not occur in the cohomology of any power of A. Thus A is not dominated by the set of powers of itself. ◻

Remark 5.2. The general Hodge conjecture implies that the Hodge structure M constructed in the above proof is supported on a divisor. Since $\dim A = 2m[k : \mathbf{Q}] > m$, the weak Lefschetz theorem implies that $H^{m-2}(Y, \mathbf{Q}) \cong H^{m-2}(A, \mathbf{Q})$ for a smooth hyperplane section Y. Since $M \subset H^m(A, \mathbf{Q})$, and we have seen that $H^{m-2}(A, \mathbf{Q})$ does not contain a Hodge substructure isomorphic to $M(1)$, we conclude that M is not supported on any smooth divisor.

Theorem 5.3. *Let A be an abelian variety satisfying the hypotheses of Proposition* 5.1. *Assume further that $\dim_{D(A)} H_1(A, \mathbf{Q}) = 4$. Then there exists an abelian variety B such that each power of A is dominated by the set of abelian varieties of the form $A^i \times B^j$.*

Proof. Let $\widetilde{G} \to G$ be the universal covering of G. Then (3.2.1) shows that $\widetilde{G}(\mathbf{R}) \cong \prod_{\alpha\in S} \widetilde{G}_\alpha$, where each $\widetilde{G}_\alpha$ is isomorphic to the real Lie group $\mathrm{Spin}(6, 2)$. Since the derivative of $\widetilde{G}(\mathbf{R}) \to G(\mathbf{R})$ is an isomorphism, we identify their Lie algebras.

Let $\mathcal{A} \to \mathcal{V}$ be a Kuga fiber variety obtained from the H_2-morphism $G \to \mathrm{Sp}(V, \beta)$. This is the Hodge family determined by A, so A is the fiber $\mathcal{A}_P$ for a general point $P \in \mathcal{V}$. Let ρ be the composite H_2-morphism

$$\widetilde{G} \to G \to \mathrm{Sp}(V, \beta).$$

Satake ([15, p. 451], [16]) also constructed an H_2-morphism

$$\rho' : \widetilde{G} \to \mathrm{Sp}(V', \beta')$$

such that the scalar extension of ρ' to $\mathbf{R}$ is equivalent to $\oplus_{\alpha \in S} \rho_\alpha$, where ρ_α is the direct sum of two copies of the spin representation of $\widetilde{G}_\alpha$ having highest weight μ_3. Let $\mathcal{B} \to \mathcal{V}$ be the Kuga fiber variety obtained from this representation, and let $B := \mathcal{B}_P$. Proposition 2.3.1 implies that the Hodge group of B is $\rho'(\widetilde{G})$. The representation of $\widetilde{G}_\alpha$ with highest weight $2\mu_3$ then appears in $H^2(B^2, \mathbf{R})$. Let M' be an irreducible Hodge substructure of $H^2(B^2, \mathbf{Q})$ containing this representation.

The representation

$$\rho \oplus \rho' : \widetilde{G} \to \mathrm{Sp}(V \oplus V', \beta \oplus \beta')$$

also satisfies the H_2-condition. The Kuga fiber variety obtained from this is the fiber product $\mathcal{A} \times_{\mathcal{V}} \mathcal{B} \to \mathcal{V}$. The fiber over the point P is $A \times B$, so Proposition 2.3.1 implies that $G(A \times B) = (\rho \oplus \rho')(\widetilde{G})$. Since M and M' are equivalent $\widetilde{G}$-modules in the cohomology of $A \times B^2$, with $M \subset H^4(A \times B^2, \mathbf{Q})$ and $M' \subset H^2(A \times B^2, \mathbf{Q})$, we have $M' \cong M(1)$. The arguments in Case 3 of the proof of [2, Theorem 5.1, pp. 352–353] then show that any power of A is dominated by the set of abelian varieties of the form $A^i \times B^j$. $\square$

Remark 5.4. The symmetric domains belonging to the hermitian groups $\mathrm{SO}^\star(8)$ and $\mathrm{Spin}(6,2)$ are isomorphic (cf. [17, Exercise 1, p. 289]). This exceptional isomorphism is the reason why the case $m = 4$ needs separate treatment. For odd m, $\operatorname{disc} T$ is not a square (it is negative), so Theorem 4.1 applies. As explained in the proof of Corollary 4.3, the case $m = 2$ is also covered by Theorem 4.1. The next theorem deals with the remaining cases.

THEOREM 5.5. *Let A be an abelian variety satisfying the hypotheses of Proposition* 5.1. *Assume further that* $\dim_{D(A)} H_1(A, \mathbf{Q}) > 4$. *Then A is not dominated by the class of all abelian varieties.*

Proof. Lemma 2.1.1 shows that $G(M)$, the Hodge group of the Hodge structure M constructed in the proof of Proposition 5.1, is a quotient of G. Since G is (almost) simple, and M is a nontrivial Hodge structure, we see that $G(M)$ is isogenous to G. Suppose $M(1)$ occurs as a Hodge substructure of $H^{m-2}(B, \mathbf{Q})$ for an abelian variety B. Clearly, the Hodge groups of M and $M(1)$ are equal. Using Lemma 2.1.1 again, we find that $G(M)$ is a quotient of $G(B)$. Therefore $\mathfrak{g}(B) = \mathfrak{g} \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is a reductive Lie algebra which acts trivially on $M(1)$. Since $\mathfrak{g}$ is simple, it follows that $\mathfrak{g}^{\mathrm{ss}}(B) = \mathfrak{g} \oplus \mathfrak{g}_2^{\mathrm{ss}}$.

Let $W := H_1(B, \mathbf{Q})$, and let γ be a Riemann form for B. As explained in §2.3, the inclusion $\rho : G^{\mathrm{ss}}(B) \to \mathrm{Sp}(W, \gamma)$ is an H_1-morphism which defines a Hodge family $\mathcal{B} \to \mathcal{W}$ having B as a fiber over a general point $Q \in \mathcal{W}$. We

have the primary decomposition $W = \oplus_\alpha W_\alpha$, where the W_α are the maximal primary $G^{\mathrm{ss}}(B)$-submodules of W (§2.4). Since $G(\mathbf{R})$ has no compact factors, Lemma 2.4.1 implies that any W_α is either trivial as a $\mathfrak{g}$-module or trivial as a $\mathfrak{g}_2^{\mathrm{ss}}$-module.

Let W_1 be the sum of those W_α's on which $\mathfrak{g}$ acts nontrivially, and let W_2 be the sum of those W_α's on which $\mathfrak{g}$ acts trivially. Then $W = W_1 \oplus W_2$, where $\mathfrak{g}_2^{\mathrm{ss}}$ acts trivially on W_1, and $\mathfrak{g}$ acts trivially on W_2. The H_1-morphisms

$$\rho_i : G^{\mathrm{ss}}(B) \to \mathrm{Sp}(W_i, \gamma | W_i \times W_i), \qquad i = 1, 2,$$

define Kuga fiber varieties $\mathcal{B}_i \to \mathcal{W}$ whose fiber product is isogenous to $\mathcal{B} \to \mathcal{W}$. We then have B isogenous to $B_1 \times B_2$, where B_i is the fiber of $\mathcal{B}_i$ over the point Q. Since $\mathcal{B} \to \mathcal{W}$ is a Hodge family, a theorem of Mumford [13, p. 348] implies that it contains a fiber of CM-type. It follows that $\mathcal{B}_i \to \mathcal{W}$ also contain CM fibers, and the same theorem of Mumford implies that they are Hodge families. Since Q is a general point of $\mathcal{W}$, we have $G^{\mathrm{ss}}(B_i) = \rho_i(G^{\mathrm{ss}}(B))$. Thus $\mathfrak{g}(B_1) = \mathfrak{g}$, and $\mathfrak{g}^{\mathrm{ss}}(B_2) = \mathfrak{g}_2^{\mathrm{ss}}$. Since $\mathfrak{g}_2$ acts trivially on $M(1)$, we see that $M(1)$ occurs in the cohomology of B_1, and we assume without loss of generality that $B = B_1$, and $G(B)$ is isogenous to G.

The isomorphism of G-modules

$$H^m(A, \mathbf{Q}) \supset M \xrightarrow{\cong} M(1) \subset H^{m-2}(B, \mathbf{Q})$$

is induced by a Hodge cycle ζ on $A \times B$. Now, $G(A \times B)$ is a subgroup of $G(A) \times G(B)$ whose projection to each factor is surjective. By Goursat's Lemma (cf. Gordon [8, Proposition B.71, p. 316]), $\mathfrak{g}(A \times B)$ is either $\mathfrak{g}(A) \times \mathfrak{g}(B)$, or the graph of an isomorphism from $\mathfrak{g}(A)$ to $\mathfrak{g}(B)$. Since ζ is invariant under $G(A \times B)$, the possibility $\mathfrak{g}(A) \times \mathfrak{g}(B)$ is ruled out, so we conclude that $G(A \times B)$ is isogenous to G.

Satake's classification of H_1-morphisms [15] shows that the inclusion of $G(A \times B)$ into the symplectic group of $H_1(A \times B, \mathbf{Q})$ satisfies the H_2-condition and is a multiple of the standard representation. Lemma 2.2.1 now implies that B is isogenous to a power of A, contradicting Proposition 5.1. □

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References

[1] S. Abdulali, Algebraic cycles in families of abelian varieties, *Canad. J. Math.* **46** (1994), 1121–1134.

[2] ———, Abelian varieties and the general Hodge conjecture, *Compositio Math.* **109** (1997), 341–355.

[3] ———, Abelian varieties of type III and the Hodge conjecture, *Internat. J. Math.* **10** (1999), 667–675.

[4] S. Abdulali, Filtrations on the cohomology of abelian varieties, in *The Arithmetic and Geometry of Algebraic Cycles* (Banff, 1998; B. B. Gordon, J. D. Lewis, S. Müller-Stach, S. Saito, and N. Yui, eds.), *CRM Proc. Lecture Notes* **24**, A. M. S., Providence, RI, 2000, pp. 3–12.

[5] ———, Hodge structures on abelian varieties of CM-type, *J. Reine Angew. Math.* **534** (2001), 33–39.

[6] P. Deligne (notes by J. S. Milne), Hodge cycles on abelian varieties, in P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, *Hodge Cycles, Motives, and Shimura Varieties*, *Lect. Notes in Math.* **900**, Springer-Verlag, New York, 1982, pp. 9–100.

[7] B. van Geemen and A. Verra, Quaternionic Pryms and Hodge classes, *Topology*, to appear.

[8] B. B. Gordon, A Survey of the Hodge Conjecture for Abelian Varieties, Appendix B to J. D. Lewis, *A Survey of the Hodge Conjecture*, Second Edition, *CRM Monogr. Ser*. **10**, A. M. S., Providence, RI, 1999, pp. 297–356.

[9] A. Grothendieck, Hodge's general conjecture is false for trivial reasons, *Topology* **8** (1969), 299–303.

[10] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, *Pure and Applied Math.* **80**, Academic Press, New York, 1978.

[11] B. J. J. Moonen and Yu. G. Zarhin, Hodge classes and Tate classes on simple abelian fourfolds, *Duke Math. J*. **77** (1995), 553–581.

[12] D. Mumford, Families of abelian varieties, in *Algebraic Groups and Discontinuous Subgroups* (Boulder, CO, 1965; A. Borel and G. D. Mostow, eds.), *Proc. Sympos. Pure Math.* **9**, A. M. S., Providence, RI, 1966, pp. 347–351.

[13] ———, A note of Shimura's paper "Discontinuous groups and abelian varieties", *Math. Ann.* **181** (1969), 345–351.

[14] V. K. Murty, Exceptional Hodge classes on certain abelian varieties, *Math. Ann.* **268** (1984), 197–206.

[15] I. Satake, Holomorphic imbeddings of symmetric domains into a Siegel space, *Amer. J. Math.* **87** (1965), 425–461.

[16] ———, Symplectic representations of algebraic groups satisfying a certain analyticity condition, *Acta Math.* **117** (1967), 215–279.

[17] ———, *Algebraic Structures of Symmetric Domains*, *Publ. Math. Soc. Japan* **14** (Kanô Mem. Lect. **4**), Iwanami Shoten, Tokyo, and Princeton Univ. Press, Princeton, NJ, 1980.

[18] C. Schoen, Addendum to: Hodge classes on self-products of a variety with an automorphism, *Compositio Math.* **114** (1998), 329–336.

[19] G. Shimura, On analytic families of polarized abelian varieties and automorphic functions, *Ann. of Math.* **78** (1963), 149–192.

[20] ———, On the field of definition for a field of automorphic functions, *Ann. of Math.* **80** (1964), 160–189.

[21] ———, Moduli of abelian varieties and number theory, in *Algebraic Groups and Discontinuous Subgroups* (Boulder, CO, 1965; A. Borel and G. D. Mostow, eds.), *Proc. Sympos. Pure Math.* **9**, A. M. S., Providence, RI, 1966, pp. 312–332.

[22] J. Tits, Classification of algebraic semisimple groups, in *Algebraic Groups and Discontinuous Subgroups* (Boulder, CO, 1965; A. Borel and G. D. Mostow, eds.), *Proc. Sympos. Pure Math.* **9**, A. M. S., Providence, RI, 1966, pp. 33–62.

[23] V. S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*, Prentice-Hall, Englewood Cliffs, NJ, 1974.

[24] A. Weil, Abelian varieties and the Hodge ring, in *Œuvres Scientifiques*: *Collected Papers*, Vol. III (1964–1978) (1977c), Springer-Verlag, New York, 1979; Corrected second printing, 1980, pp. 421–429.

(Received December 19, 2000)